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On the stability of stratified viscous plane Couette flow. Part 1. Constant buoyancy frequency

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This paper is concerned with a general study of the modal structure for stratified viscous plane Couette flow with a constant buoyancy frequency. When the overall Richardson number Ri is zero, the velocity and temperature modes are distinct but as Ri is increased there is an intricate interaction between them. Some simple analytical results are obtained for large and small values of the Reynolds number and more detailed results are given for $Ri = 0, \frac{1}{8}, \frac{1}{4}$ and $\frac{1}{2}$. The present theory would appear to be reasonably complete for $0 \leqslant Ri \leqslant \frac{1}{4}$; for $Ri > \frac{1}{4}$, however, an important open question concerns the relationship between the limiting form of the viscous modes as the Reynolds number tends to infinity and the spectrum of internal gravity waves.

1. Introduction

The effects of stratification on the stability of parallel shear flows are usually studied in an inviscid approximation (see, for example, Drazin & Howard 1966; Howard & Maslowe 1973). The problem is then governed by the Taylor–Goldstein equation, which, like Rayleigh's stability equation for a homogeneous fluid, is singular at any point where the basic flow speed $U(z)$ is equal to the wave speed c. A statically stable density distribution would be expected, on general physical grounds, to have a stabilizing effect and it has been proved by Miles (1961) and Howard (1961) that stability is assured if the local Richardson number $J(z) = -g\rho'/\rho U'^2$ exceeds $\frac{1}{4}$ everywhere. Circumstances are known, however, in which a basic flow which is stable in the absence of stratification can become unstable when certain statically stable density distributions are imposed, and some examples of this phenomenon have been discussed by Howard & Maslowe (1973). A particularly dramatic example of this phenomenon was discovered recently by Huppert (1973, §2.3), who showed that plane Couette flow with (dimensionless) buoyancy frequency $N^2(z) = z^2$ is unstable if the (overall) Richardson number is larger than $\frac{1}{4}$.

Huppert's results for the inviscid problem are so striking that we felt it would be of interest to study the effects of including both viscosity and thermal conductivity. Before doing so, however, we felt that it would be desirable, for purposes of comparison, to consider the simpler problem of stratified plane Couette flow with a constant buoyancy frequency. This is a problem for which no instability is to be expected (and none was found) but it does lead to an interesting and basic study of the interaction between the velocity and temperature modes in a stratified shear flow. In part 2 we shall consider Huppert's problem and study the changes in the modal structure which lead to instability even when the stratification is statically stable.

2. The governing equations

The linearized disturbance equations which govern the stability of thermally stratified shear flows were first derived by Koppel (1964). In deriving these equations the usual Boussinesq approximation was made and the effects of both viscosity and thermal conductivity were included. Koppel also derived the generalization of Squire's theorem for this problem and showed that "the three-dimensional problem is equivalent to a two-dimensional problem at a smaller Reynolds number and a larger Richardson number". In the present paper, therefore, we shall consider only two-dimensional disturbances. The corresponding results for three-dimensional disturbances could, if required, then be obtained by the use of Squire's transformation as discussed by Gage & Reid (1968).

A slightly different form for the governing equations has been given by Baldwin & Roberts (1970), who considered the perturbation σ_* in the buoyancy force per unit mass rather than the perturbation θ_* in the temperature. These two quantities are, of course, related by $\sigma_* = \alpha g \theta_*$, where α is the coefficient of thermal expansion and g is the acceleration due to gravity, but the equations obtained by Baldwin & Roberts have some advantages, particularly in relation to the inviscid theory.

For the purpose of writing the governing equations in dimensionless form it is convenient to introduce a characteristic length L_* equal to half the width of the channel and a characteristic velocity U_* equal to the maximum velocity of the basic flow. In addition, σ_* will be made dimensionless with respect to g. We now introduce a stream function $\phi(z)\, e^{i\alpha(x-ct)}$ in the usual way so that $u = \phi'$ and $w = -i\alpha\phi$, and let σ also be of the form $\sigma(z)\, e^{i\alpha(x-ct)}$. The governing equations can then be written in the dimensionless form

$$L_4\phi = Ri\,\sigma, \quad L_2\sigma = -N^2(z)\,\phi, \tag{2.1a, b}$$

where

$$L_2 = (i\alpha RP)^{-1}(D^2-\alpha^2)-(U-c) \tag{2.2}$$

and

$$L_4 = (i\alpha R)^{-1}(D^2-\alpha^2)^2-(U-c)(D^2-\alpha^2)+U''. \tag{2.3}$$

We also have the boundary conditions

$$\phi = \phi' = \sigma = 0 \quad \text{at} \quad z = \pm 1. \tag{2.4}$$

The Prandtl number P, Reynolds number R and (overall) Richardson number Ri which appear in these equations are given by

$$P = \nu/\kappa, \quad R = U_*L_*/\nu, \quad Ri = gL_*/U_*^2, \tag{2.5}$$

and the unperturbed flow is characterized by the velocity profile $U(z)$ and buoyancy frequency $N(z)$, where

$$N^2(z) = -\frac{L_*}{\rho_*}\frac{d\rho_*}{dz_*}. \tag{2.6}$$

For some purposes it is convenient to eliminate σ between (2.1a, b) and this gives

$$L_2L_4\phi = -Ri\,N^2(z)\,\phi, \tag{2.7}$$

together with the boundary conditions

$$\phi = \phi' = L_4\phi = 0 \quad \text{at} \quad z = \pm 1. \tag{2.8}$$

If we formally let $R \to \infty$ in (2.7) we obtain the Taylor–Goldstein equation

$$(U-c)(D^2-\alpha^2)\phi - U''\phi + Ri(U-c)^{-1}N^2(z)\phi = 0, \tag{2.9}$$

which governs the stability of stratified inviscid shear flows. Equation (2.9) plays the same role in the present theory as Rayleigh's equation does in the stability theory for homogeneous shear flows. In the inviscid theory, the *local* Richardson number

$$J(z) = -\frac{g}{\rho_*}\frac{d\rho_*/dz_*}{(dU_*/dz_*)^2} \tag{2.10}$$

plays a particularly important role; for, as Miles (1961) and Howard (1961) have proved, a *sufficient* condition for stability is that $J(z)$ should exceed $\frac{1}{4}$ everywhere. It is important, therefore, to distinguish between the local and overall Richardson numbers, which are related by

$$J(z) = Ri\,N^2(z)/\{U'(z)\}^2. \tag{2.11}$$

A general discussion of (2.1) or (2.7) is clearly a formidable task and we now wish to consider various simplifying assumptions. When asymptotic methods of approximation are used, an enormous simplification can be achieved by assuming that $P = 1$. This assumption was made by Gage & Reid (1968) in their study of the stability of stratified plane Poiseuille flow for which $U(z) = 1-z^2$ and $N^2(z) = 1$. More recently, Tveitereid (1974) has considered the same problem but without assuming that $P = 1$ and he found that his results were not sensitive to variations in the Prandtl number over a fairly wide range. This work by Gage & Reid and Tveitereid was concerned primarily with the determination of the value of Ri at which the flow becomes completely stable and, as a result, only one mode was studied. If, however, one is interested in the effects of stratification on the general structure of the modes then it would be natural to consider the simpler case of plane Couette flow with $U(z) = z$ and $N^2(z) = 1$. The modal structure for unstratified plane Couette flow is now fairly well understood from the work of Grohne (1954), Gallagher & Mercer (1962, 1964) and Gallagher (1974). This work shows that most of the essential features of the problem are still present if we set $\alpha = 0$ but treat αR as finite. This approximation not only reduces the amount of numerical work required but also leads to a further simplification in the asymptotic parts of the analysis.

3. The velocity and temperature modes when $Ri = 0$

When $Ri = 0$ the modal structure has a particularly simple form. In this limit (2.1a, b) become

$$L_4\phi = 0, \quad L_2\sigma = -N^2(z)\phi, \tag{3.1}$$

and we can therefore distinguish two classes of modes: the *velocity modes*, which satisfy the Orr–Sommerfeld equation $L_4\phi = 0$, and the *temperature modes*, for which $\phi \equiv 0$ and $L_2\sigma = 0$. These modes are thus independent of the stratification and depend only on the velocity distribution of the basic flow. The velocity modes for plane Couette flow have been extensively studied and only a brief summary of the most essential results

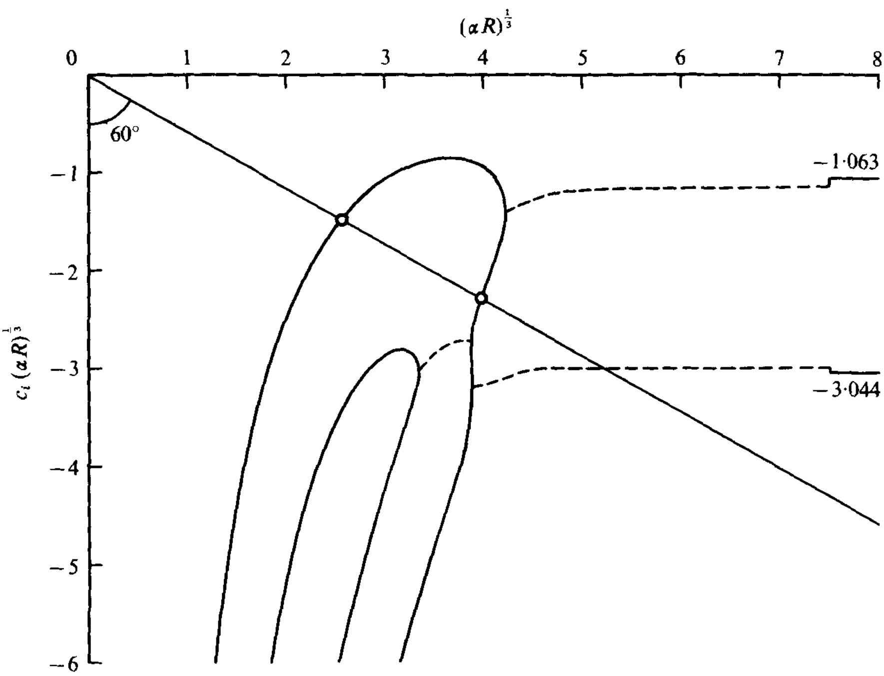


FIGURE 1. The first four velocity modes for plane Couette flow with $\alpha = 0$ but αR finite. ——, symmetric modes with $c_r = 0$; – – –, asymmetric modes with $c_r \neq 0$; ○, exact analytical results.

need be given here. Simple analytical results can be obtained as $\alpha R \to 0$ or ∞, and asymptotic methods are effective in dealing with the so-called 'symmetric' (or 'standing' or 'centre') modes, for which $c_r = 0$. The corresponding analysis for the temperature modes is very much simpler since they satisfy a second-order equation but the results do have a certain qualitative similarity with the velocity modes and this suggests that a more general study of the temperature modes and their relationship to the velocity modes would be of considerable interest.

The velocity modes

Consider first the limit as $\alpha R \to 0$, which corrresponds to considering the decay modes for a fluid at rest. In this limit $c_r = 0$ and $\alpha R c_i \to -\lambda^2$. When $\alpha = 0$, λ is determined from the eigenvalue problem $\phi^{iv} + \lambda^2\phi'' = 0$ with $\phi = \phi' = 0$ at $z = \pm 1$. Thus, for the even modes we have $\sin\lambda = 0$ and hence $\lambda/\pi = 1, 2, 3, \ldots$; for the odd modes, however, we have $\tan\lambda = \lambda$ and hence $\lambda/\pi \simeq 1{\cdot}430,\ 2{\cdot}459,\ 3{\cdot}471, \ldots$.

Consider next the limit as $\alpha R \to \infty$. In this limit the modes are of the 'asymmetric' (or 'travelling' or 'edge') type with

$$(1 \pm c_r + ic_i)(\alpha R)^{\frac{1}{3}} \to -z_{\pm s}\, e^{-\frac{1}{6}\pi i}, \tag{3.2}$$

where $z_{\pm s}$ $(s = 1, 2, \ldots)$ are the (complex) zeros of $A_1(z, 1)$ with $\mathrm{Im}(-z_s) > 0$, $z_{-s} = z_s^*$ and, following Reid (1972),

$$A_1(z, 1) = -\int_z^\infty \mathrm{Ai}(t)\, dt. \tag{3.3}$$

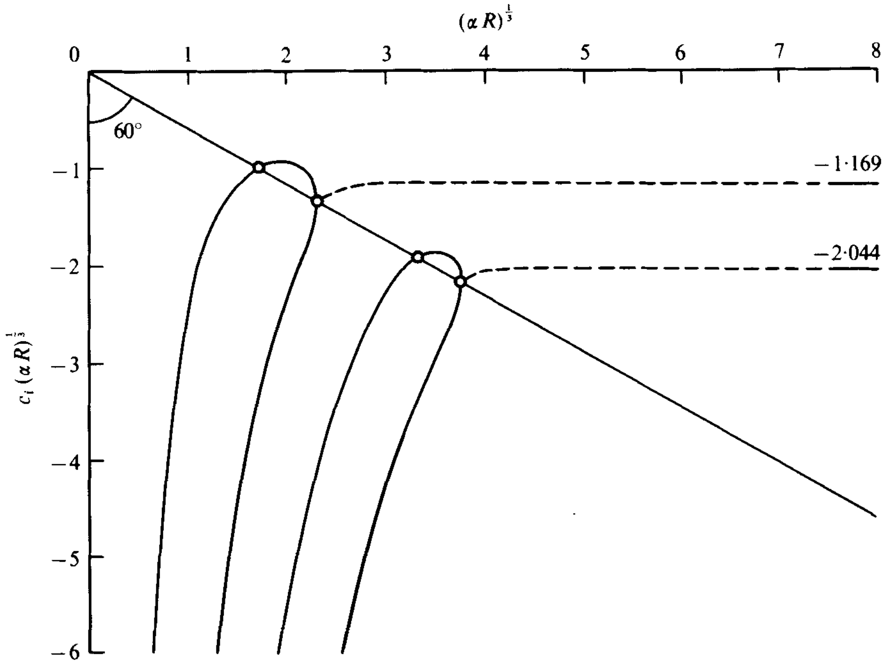


FIGURE 2. The first four temperature modes for plane Couette flow with $\alpha = 0$ but αR finite.

For $s = 1$ we have $z_1 e^{-\frac{1}{6}\pi i} \simeq -4\cdot1288 + 1\cdot0626i$ and for $s \geqslant 2$ it is sufficient for most purposes to use an asymptotic approximation for the zeros such as the one given by Reid (1974). In terms of the notation introduced in the appendix, $z_s \equiv \alpha_s(1)$.

For finite values of αR the symmetric modes can be calculated with good accuracy by the use of asymptotic methods but for the asymmetric modes a direct numerical calculation seems to be required. The velocity modes appear in groups of four interconnected modes and the behaviour of the first such group is shown in figure 1. Additional results have been given by Gallagher (1974), who calculated the first twelve modes for $\alpha = 0$, $0\cdot6$, 1 and 2.

The temperature modes

In discussing the temperature modes it is not necessary to set $P = 1$ or $\alpha = 0$ and we do so only to simplify the comparison with the corresponding results for the velocity modes. Thus, as $\alpha R \to 0$, we again have $c_r = 0$ and $\alpha R c_i \to -\mu^2$ (say), where μ is now determined from the eigenvalue problem $\sigma'' + \mu^2\sigma = 0$ with $\sigma = 0$ at $z = \pm 1$. For the even modes we have $\cos\mu = 0$ and hence $\mu/\pi = \frac{1}{2}, \frac{3}{2}, \ldots$; for the odd modes we have $\sin\mu = 0$ and hence $\mu/\pi = 1, 2, \ldots$. The even velocity modes and the odd temperature modes therefore have the same eigenvalues but the corresponding eigenfunctions are different.

More generally, the eigenvalue problem for the temperature modes is defined by

$$L_2\sigma = 0 \quad \text{with} \quad \sigma = 0 \quad \text{at} \quad z = \pm 1. \tag{3.4}$$

If this equation is multiplied by σ^* and then integrated over the range of z we obtain

$$c\int_{-1}^{1} |\sigma|^2 dz = \int_{-1}^{1} U(z)\,|\sigma|^2 dz + (i\alpha RP)^{-1}\int_{-1}^{1} \{|D\sigma|^2 + \alpha^2|\sigma|^2\}\,dz, \tag{3.5}$$

from which it immediately follows that

$$U_{\min} < c_r < U_{\max}, \quad c_i < -(\tfrac{1}{4}\pi^2 + \alpha^2)/\alpha RP. \tag{3.6}$$

This estimate for c_i is realistic for small but not for large values of αRP. It does show, however, that the temperature modes are all stable.

The general solution of (3.4) can be expressed in terms of the Airy functions $A_1(\zeta + \alpha^2\epsilon^2)$ and $A_2(\zeta + \alpha^2\epsilon^2)$, where

$$\zeta = (z-c)/\epsilon, \quad \epsilon = (i\alpha R)^{-\frac{1}{3}}, \tag{3.7}$$

and we have again set $P = 1$. The case $P \neq 1$ can, of course, be treated by merely redefining ϵ as $(i\alpha RP)^{-\frac{1}{3}}$. To simplify the discussion we shall now suppose that $\alpha = 0$ with αR finite. The eigenvalue relation then becomes

$$\Delta(\zeta_1, \zeta_2) \equiv A_1(\zeta_1) A_2(\zeta_2) - A_1(\zeta_2) A_2(\zeta_1) = 0, \tag{3.8}$$

where $\zeta_1 = -(1+c)/\epsilon$ and $\zeta_2 = (1-c)/\epsilon$. The roots of this equation are of two distinct types depending upon whether $c_r = 0$ or not. When $c_r = 0$ the roots are symmetrically located with respect to the ray $\mathrm{ph}\,\zeta = \frac{2}{3}\pi$ and for this reason they are sometimes called the symmetric modes; otherwise they are of asymmetric type.

In discussing the symmetric modes it is convenient to introduce the polar representation

$$\zeta_1 = r\exp(\tfrac{2}{3}\pi i + \theta i), \quad \zeta_2 = r\exp(\tfrac{2}{3}\pi i - \theta i), \tag{3.9}$$

in terms of which we have

$$(\alpha R)^{\frac{1}{3}} = r\sin\theta, \quad c = -i\cot\theta. \tag{3.10}$$

When ζ_1 and ζ_2 lie on the anti-Stokes lines $\mathrm{ph}\,\zeta = \pi$ and $\frac{1}{3}\pi$ respectively, i.e. when $\theta = \frac{1}{3}\pi$, an exact solution of (3.6) is possible. In this special case we have

$$\zeta_1 = -r, \quad \zeta_2 = re^{\frac{1}{3}\pi i}, \quad (\alpha R)^{\frac{1}{3}} = \tfrac{1}{2}\sqrt{3}\,r, \quad c = -i\tfrac{1}{3}\sqrt{3}. \tag{3.11}$$

After some reduction, (3.8) becomes

$$\Delta = \tfrac{1}{4}e^{\frac{2}{3}\pi i}\{3\,\mathrm{Ai}^2(-r) - \mathrm{Bi}^2(-r)\}, \tag{3.12}$$

which can also be written in the form

$$\Delta = \tfrac{1}{3}e^{\frac{2}{3}\pi i}\, rJ_{-\frac{1}{3}}(\xi)\, J_{\frac{1}{3}}(\xi), \tag{3.13}$$

where $\xi \equiv \frac{2}{3}r^{\frac{3}{2}}$. The zeros of this equation are thus simply $j_{\pm\frac{1}{3},s}$ $(s = 1, 2, \ldots)$. For most purposes, however, it is entirely adequate to use the asymptotic approximations $j_{\pm\frac{1}{3},s} \sim (s \pm \frac{1}{6} - \frac{1}{4})\pi$.

If the Airy functions which appear in (3.8) are now approximated in the complete sense by the leading terms of their asymptotic expansions, then we obtain

$$\Delta \sim \tfrac{1}{2}\pi^{-1} e^{\frac{2}{3}\pi i} r^{-\frac{1}{2}}\{\sin(2\xi\sin\tfrac{3}{2}\theta) + \tfrac{1}{2}\exp(-2\xi\cos\tfrac{3}{2}\theta)\}. \tag{3.14}$$

When $\theta = \frac{1}{3}\pi$ this result reduces to the corresponding expansion of (3.13). In the derivation of the approximation (3.14) it was assumed not only that r is large but also that θ is fixed, i.e. that αR is large. Nevertheless, it can easily be verified that if we let $\theta \to 0$ in (3.10) and (3.14) then we recover the results obtained above for $\alpha R \to 0$. For computational purposes, therefore, the approximation (3.14) is entirely adequate.

The direct calculation of the asymmetric modes from (3.8) or, more precisely, from an approximation like (3.14) is more difficult. As $\alpha R \to \infty$, however, it is easy to show that

$$(1 \pm c_r + ic_i)(\alpha R)^{\frac{1}{3}} \to -a_s e^{-\frac{1}{6}\pi i}, \tag{3.15}$$

where a_s ($s = 1, 2, \ldots$) are the (real) zeros of Ai(z). Again, it is sufficient for most purposes to use the approximation $a_s \sim -[\frac{3}{8}\pi(4s-1)]^{\frac{2}{3}}$.

These results for the temperature modes are analogous to the results given by Reid (1974) for the velocity modes. The behaviour of the first four temperature modes is shown in figure 2. Although they differ quantitatively from the velocity modes, their general structure is remarkably similar.

4. The modal structure when $0 < Ri \leqslant \frac{1}{4}$

We now wish to consider the interaction between the velocity and temperature modes which occurs when $Ri > 0$. For fixed values of Ri, simple analytical results can be obtained as $\alpha R \to 0$ and $\alpha R \to \infty$; for finite values of αR, however, a direct numerical attack seems to be required.

Consider first the low Reynolds number limit. As $\alpha R \to 0$ it is known that the wave speed c can be expanded in the form

$$i\alpha Rc = c^{(0)} + i\alpha Rc^{(1)} + (i\alpha R)^2 c^{(2)} + \ldots, \tag{4.1}$$

where $c^{(0)}$ and $c^{(1)}$ give the limiting values of c_i and c_r respectively. More generally, however, it is not difficult to show that $c^{(0)}$ is independent of both U and N^2, that $c^{(1)}$ depends only on U, and that $c^{(2)}$ depends on both U and N^2. Thus, in this limit, we can continue to distinguish between the velocity and temperature modes even when $Ri > 0$.

Consider next the high Reynolds number limit. General results are difficult to obtain in this limit and we shall therefore simplify the problem drastically by assuming that

$$P = 1, \quad U(z) = z, \quad N^2(z) = 1, \quad \alpha = 0 \quad \text{with } \alpha R \text{ finite.} \tag{4.2}$$

With these assumptions, (2.7) can be reduced to a form which contains only the single parameter Ri. For this purpose let

$$\epsilon = (i\alpha R)^{-\frac{1}{3}}, \quad \zeta = (z-c)/\epsilon, \quad D = d/d\zeta, \quad A = D^2 - \zeta. \tag{4.3}$$

Equation (2.7) then becomes simply

$$(A^2D^2 + Ri)\phi = 0 \tag{4.4}$$

and the boundary conditions are

$$\phi = D\phi = AD^2\phi = 0 \quad \text{at} \quad \zeta = \zeta_1, \zeta_2, \tag{4.5}$$

where $\zeta_1 = -(1+c)/\epsilon$ and $\zeta_2 = (1-c)/\epsilon$. From the discussion given by Hughes & Reid (1968) in connexion with a related problem it follows immediately that (4.4) can be written in the factorized form

$$(AD + p_1)(AD + p_2)\phi = 0, \tag{4.6}$$

where p_1 and p_2 are the roots of the indicial equation $p(p-1) + Ri = 0$ associated with the Taylor–Goldstein equation (2.9), i.e.

$$p_1, p_2 = \tfrac{1}{2}\{1 \pm (1 - 4Ri)^{\frac{1}{2}}\}. \tag{4.7}$$

Thus, provided $p_1 \neq p_2$, i.e. $Ri \neq \frac{1}{4}$, the solutions of (4.6) can be expressed in terms of the generalized Airy functions

$$A_1(\zeta, p_i + 1), \quad A_2(\zeta, p_i + 1), \quad B_3(\zeta, p_i + 1) \quad (i = 1, 2), \tag{4.8}$$

where

$$A_k(\zeta, p) = \frac{1}{2\pi i} \int_{L_k} t^{-p} \exp(\zeta t - \tfrac{1}{3}t^3)\, dt, \tag{4.9}$$

$$B_k(\zeta, p) = \frac{1}{2\pi i} \int_{\infty \exp[(k-1)\pi i]}^{0+} t^{-p} \exp(\zeta t - \tfrac{1}{3}t^3)\, dt, \tag{4.10}$$

and L_k are the usual Airy contours with $k = 1, 2, 3$. Since the integrands in (4.9) and (4.10) are, in general, multiple-valued, we shall suppose that a cut has been introduced in the t plane running from the origin to infinity along the positive real axis. This class of Airy functions was first introduced by Hughes & Reid (1968) and has been discussed further, for integral values of p, by Reid (1972).

The required eigenvalue relation can be derived in the usual way by forming a linear combination of the six functions (4.8) and then applying the boundary conditions (4.5). In applying the third of these boundary conditions, we may note that

$$AD^2 A_k(\zeta, p+1) = -(p-1) A_k(\zeta, p) \tag{4.11}$$

with a similar relation for $B_k(\zeta, p+1)$. Thus far no restriction has been placed on the magnitude of αR. Since we are primarily interested in obtaining the generalization of the results (3.2) and (3.15) for $Ri > 0$, we shall not pause to give an analytical discussion of the symmetrical modes but proceed directly to a discussion of the asymmetrical modes for which $c_r \to \pm 1$ as $\alpha R \to \infty$. In this limit it is found that

$$\left.\begin{array}{llll} \textit{either} & \zeta_1 \to \zeta_s, & c_r \downarrow -1 & \text{as} \quad \zeta_2 \to \infty \text{ in } \mathbf{S}_1 \\ \textit{or} & \zeta_2 \to \zeta_s e^{-\frac{4}{3}\pi i}, & c_r \uparrow +1 & \text{as} \quad \zeta_1 \to \infty \text{ in } \mathbf{S}_2, \end{array}\right\} \tag{4.12}$$

which is equivalent to (3.2) with $z_{\pm s}$ replaced by ζ_s, where the $\zeta_s(Ri)$ $(s = 1, 2, \ldots)$ are the roots of a certain transcendental equation which will now be derived. It is sufficient, therefore, to consider only the first of these possibilities. The eigenvalue determinant can be substantially simplified by observing that if $|\zeta_2| \gg |\zeta_1|$ then, with an exponentially small error, we can neglect $A_1(\zeta_2, p)$ compared with $A_1(\zeta_1, p)$ and $A_2(\zeta_1, p)$ compared with $A_2(\zeta_2, p)$; we cannot, however, then neglect $A_1(\zeta_1, p)$ compared with $A_2(\zeta_2, p)$, for this would lead to a null result. In this approximation, therefore, we obtain

$$\begin{aligned} \Delta(\zeta_1, \zeta_2) \equiv{} & \frac{(p_2 - p_1)^2}{\Gamma(p_1 + 1)\,\Gamma(p_2 + 1)} \\ & \times \{\exp(2p_1 \pi i)\, A_1(\zeta_1, -p_1)\, A_1(\zeta_1, p_2)\, A_2(\zeta_2, p_1)\, A_2(\zeta_2, -p_2) \\ & - \exp(2p_2 \pi i)\, A_1(\zeta_1, p_1)\, A_1(\zeta_1, -p_2)\, A_2(\zeta_2, -p_1)\, A_2(\zeta_2, p_2)\} = 0, \end{aligned} \tag{4.13}$$

where use has been made of the second-order Wronskian relations

$$\left.\begin{aligned} W_2\{A_1(\zeta, p+1), B_3(\zeta, p+1)\} &= -\frac{e^{-2p\pi i}}{\Gamma(p+1)} A_1(\zeta, -p) \\ \text{and} \qquad W_2\{A_2(\zeta, p+1), B_3(\zeta, p+1)\} &= -\frac{e^{-4p\pi i}}{\Gamma(p+1)} A_2(\zeta, -p). \end{aligned}\right\} \tag{4.14}$$

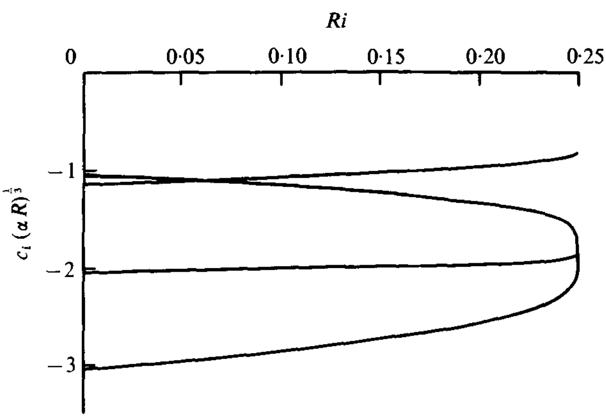


FIGURE 3. The limiting behaviour of $c_i(\alpha R)^{\frac{1}{3}}$ as $\alpha R \to \infty$ for the first four modes and $0 \leqslant Ri \leqslant \frac{1}{4}$.

This approximation can now be further simplified by replacing $A_2(\zeta_2, p)$ with the leading term of its asymptotic expansion. Thus, as $\zeta_2 \to \infty$ in $\mathbf{S}_1$, we have

$$\Delta(\zeta_1,\zeta_2) \sim \frac{(p_2-p_1)^2}{4\pi\,\Gamma(p_1+1)\,\Gamma(p_2+1)} \exp\left(-2p_2\pi i\right)\zeta_2^{-p_2} \exp\left(\tfrac{4}{3}\zeta_2^{\frac{3}{2}}\right)$$
$$\times \{A_1(\zeta_1,p_1)\,A_1(\zeta_1,-p_2) - \exp\left[2(p_2-p_1)\pi i\right]\zeta_2^{p_2-p_1} A_1(\zeta_1,-p_1)\,A_1(\zeta_1,p_2)\}. \tag{4.15}$$

Thus, when $\zeta_2 \to \infty$ in $\mathbf{S}_1$ with $0 \leqslant Ri < \frac{1}{4}$, $\zeta_1 \to \zeta_s$, where the ζ_s are the roots of the equation

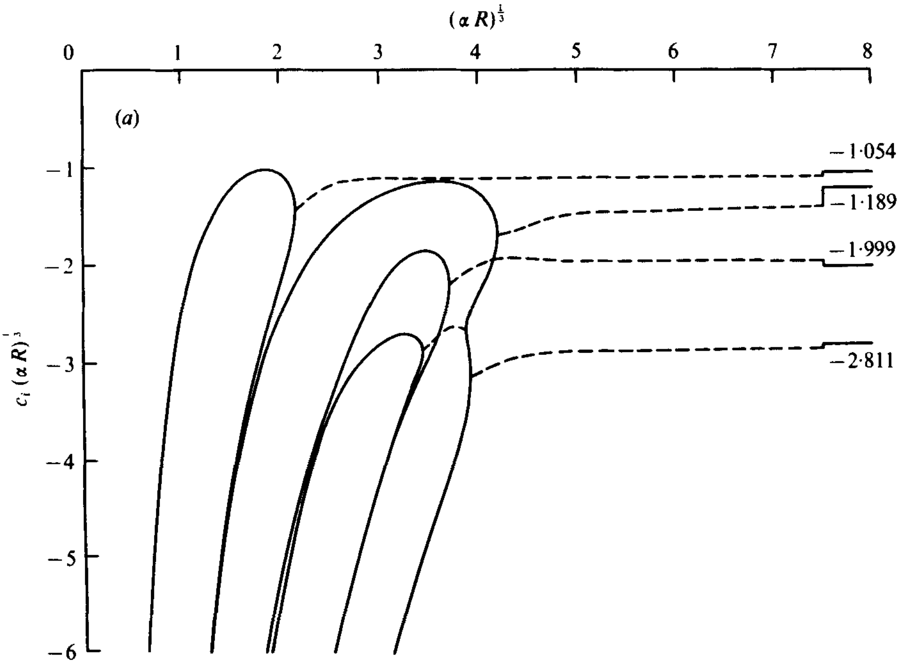
$$A_1(\zeta,p_1)\,A_1(\zeta,-p_2) = 0. \tag{4.16}$$

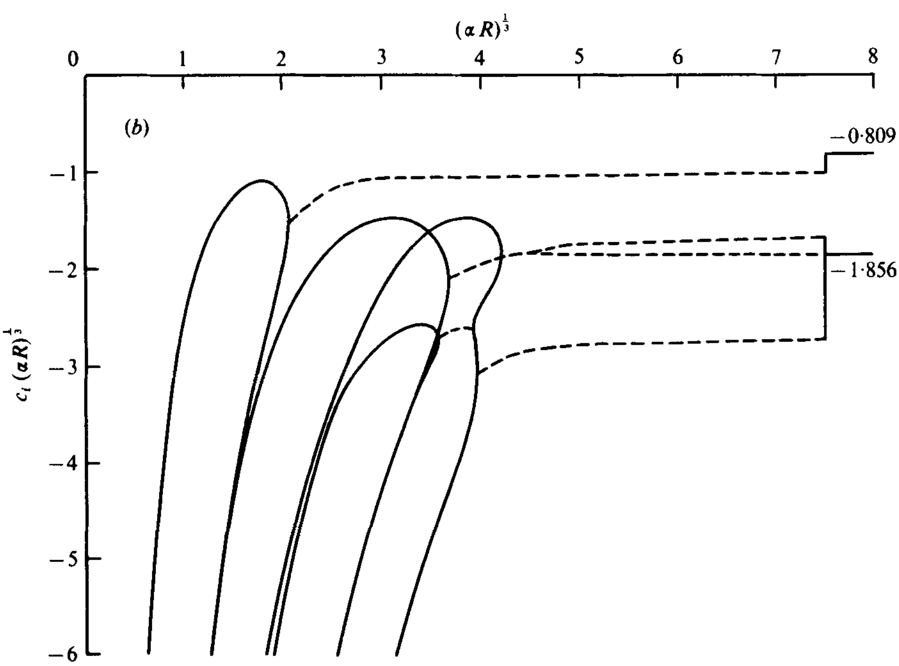
When $Ri = 0$, $p_1 = 1$ and $p_2 = 0$, and we therefore recover the previous results (3.2) and (3.15) for the velocity and temperature modes respectively. For $Ri > 0$, the roots of (4.16) must be considered in groups of four and, from the results given in the appendix, it then follows that the limiting values of $c_i(\alpha R)^{\frac{1}{3}}$ as $\alpha R \to \infty$ for the first group of modes have the behaviour shown in figure 3. The coalescence of three of these modes as $Ri \uparrow \frac{1}{4}$ is simply a consequence of the fact, as discussed in the appendix, that

$$\left.\begin{aligned} &A_1(\zeta,+\tfrac{1}{2}) = -i2^{\frac{2}{3}}\pi^{\frac{1}{2}}\,\mathrm{Ai}^2(x)\\ \text{and}\qquad &A_1(\zeta,-\tfrac{1}{2}) = -i2\pi^{\frac{1}{2}}\,\mathrm{Ai}(x)\,\mathrm{Ai}'(x),\end{aligned}\right\} \tag{4.17}$$

where $x = 2^{-\frac{2}{3}}\zeta$. It should be emphasized, however, that the approximation (4.16) to the eigenvalue relation is not uniformly valid as $Ri \downarrow \frac{1}{4}$. Nevertheless, it does provide a useful guide to the behaviour of the modes for large values of αR as Ri increases from 0 to $\frac{1}{4}$.

To obtain the complete modal structure for finite values of αR a direct numerical approach was adopted. All of the calculations were done on an IBM 360/67 computer using the method of orthonormalization due to Godunov (1961) as described by Conte (1966) and later used by Wazzan, Okamura & Smith (1968). The results are shown in figures 4(*a*), (*b*) and (*c*) for three typical values of the Richardson number: $\frac{1}{8}$, $\frac{1}{4}$ and $\frac{1}{2}$. The modes must now be considered in groups of eight, the structure of which shows substantial changes as Ri is increased. In particular it is no longer possible to maintain





FIGURES 4 (a, b). For legend see facing page.

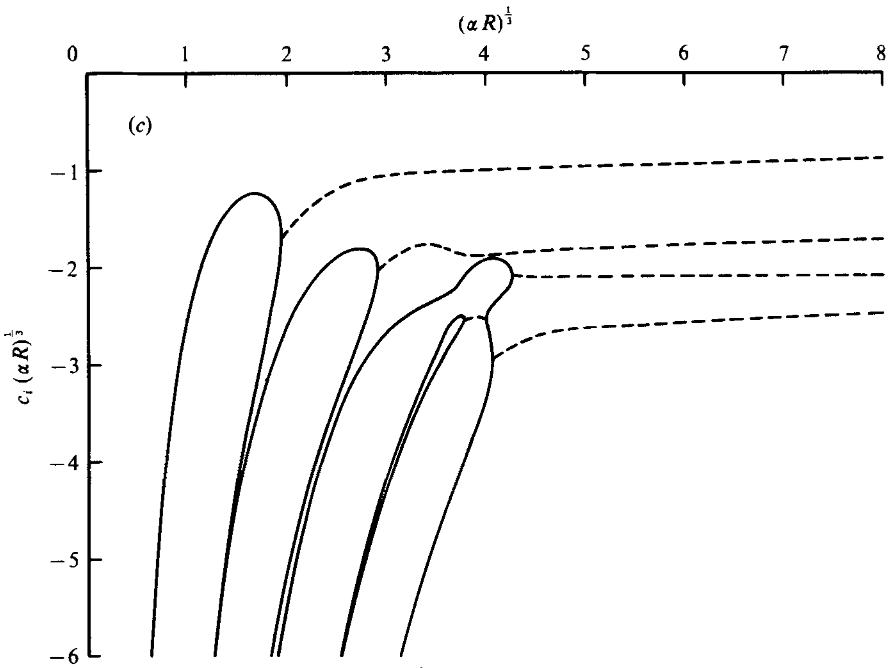


FIGURE 4. The modal structure for (*a*) $Ri = \frac{1}{8}$, (*b*) $Ri = \frac{1}{4}$ and (*c*) $Ri = \frac{1}{2}$.

a clear distinction between the velocity and temperature modes except for small and large values of αR.

It should also be mentioned that the modes found here for $0 \leqslant Ri < \frac{1}{4}$ do not have an inviscid limit throughout the closed interval $-1 \leqslant z \leqslant 1$. This is entirely to be expected, of course, in view of the results obtained by Eliassen, Høiland & Riis (1953), who showed that when $0 \leqslant Ri < \frac{1}{4}$ the corresponding inviscid problem has no discrete eigenvalues but it does have two continuous spectra, one associated with the velocity field and the other with the temperature field.

5. The internal gravity waves for $Ri > \frac{1}{4}$

When $Ri > \frac{1}{4}$, the analysis of the modal structure as $\alpha R \to \infty$ becomes substantially more difficult and we have not attempted a detailed study of this case. It is of some interest, however, to consider the relationship between the present results and those obtained by Høiland (1953) and Eliassen *et al.* (1953) for the corresponding inviscid problem. Thus, on setting $U(z) = z$ and $N^2(z) = 1$ in the Taylor–Goldstein equation (2.9), we see that the inviscid problem is given by

$$(z-c)^2(D^2-\alpha^2)\phi + Ri\,\phi = 0 \quad \text{with} \quad \phi(\pm 1) = 0. \tag{5.1}$$

When $\alpha = 0$ Høiland (1953) has shown that an explicit solution is possible and that it leads to a denumerably infinite set of real eigenvalues which represent stable internal gravity waves with wave speeds lying outside the range of $U(z)$. Thus, if c is real and $\phi(z, c)$ is an eigenfunction of the problem (5.1) then so too is $\phi(-z, -c)$. It is sufficient, therefore, to take c positive. For $\alpha = 0$, the eigenvalues are given by

$$c_n(Ri) = \coth(n\pi/2\mu) \quad (n = 1, 2, \ldots), \tag{5.2}$$

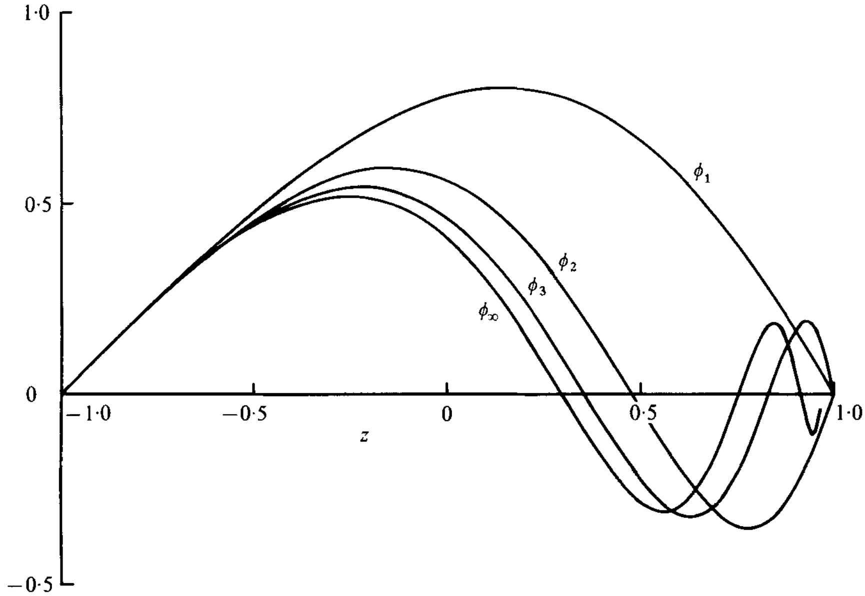


FIGURE 5. The eigenfunctions (5.3) for the internal gravity waves for $\alpha = 0$ and $Ri = 9{\cdot}25$, i.e. $\mu = 3$.

where $\mu = (Ri - \frac{1}{4})^{\frac{1}{2}}$, and the corresponding eigenfunctions, normalized such that $\phi_n'(-1) = 1$, are

$$\phi_n(z) = (-1)^{n-1}\mu^{-1}(c_n+1)^{\frac{1}{2}}(c_n-z)^{\frac{1}{2}}\sin\left\{\mu\log\left(\frac{c_n-z}{c_n-1}\right)\right\}. \tag{5.3}$$

The eigenvalues (5.2) form a decreasing sequence with limit point at 1 and for $Ri = \frac{1}{2}$, for example, all lie in the range $1 < c_n \leqslant \coth\pi \simeq 1{\cdot}0037$.

Eliassen *et al.* (1953) assumed, but were unable to prove, that this set of eigenfunctions is complete and hence, at least by implication, that there is no continuous spectrum if $Ri > \frac{1}{4}$. They then obtained a formal solution of the initial-value problem in terms of these eigenfunctions from which they concluded that the flow is stable if $Ri > \frac{1}{4}$. The eigenvalue problem (5.1) is not of Sturm–Liouville type and, as a result, the eigenfunctions (5.3) have some rather unusual properties. Consider, for example, the distribution of their zeros. For this purpose let z_{mn} $(m = 0, 1, 2, \ldots, n)$ denote the zeros of $\phi_n(z)$ with

$$-1 = z_{nn} < z_{n+1,n} < \ldots < z_{1n} < z_{0n} = 1. \tag{5.4}$$

Then it is easy to show that

$$z_{mn} = c_n - e^{m\pi/\mu}(c_n - 1). \tag{5.5}$$

In particular, the eigenfunctions $\phi_n(z)$ have no zeros in the interval $-1 < z < z_{n+1,n}$. As $n \to \infty$ or, more precisely, when $n \gg \mu$, we have

$$z_{n-1,n} \sim 1 - 2e^{-\pi/\mu} \tag{5.6}$$

and
$$\phi_n(z) \sim 2^{\frac{1}{2}}\mu^{-1}(1-z)^{\frac{1}{2}}\sin\{\mu\log[2/(1-z)]\} \quad (-1 \leqslant z < 1), \tag{5.7}$$

and both of these results are independent of n. When $Ri = \frac{1}{2}$, for example, all of the zeros lie between 0·9963 and 1. Thus, although the eigenfunctions (5.3) are certainly

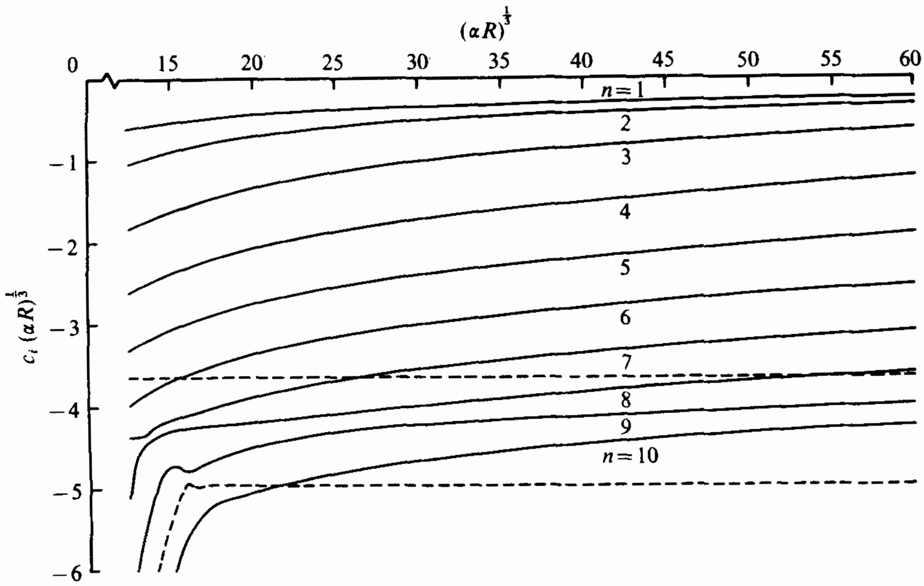


FIGURE 6. The modal structure for $Ri = 10$. ——, modes which approach internal gravity waves as $\alpha R \to \infty$ with c_r given by (5.2); – – –, modes for which c_r remains within the range of $U(z)$.

linearly independent mathematically, they are virtually indistinguishable numerically when $\frac{1}{4} < Ri \leqslant \frac{1}{2}$. To illustrate the behaviour of these eigenfunctions, a very much larger value of the Richardson number must be chosen and the results shown in figure 5 for $Ri = 9{\cdot}25$, i.e. $\mu = 3$, are fairly typical. This behaviour of the zeros of $\phi_n(z)$ may also be related to the fact that the eigenfunctions of the problem (5.1) satisfy a rather unusual orthogonality relation.

The fundamental question, however, is whether or not there is a one-to-one correspondence between the eigenvalues and eigenfunctions of the viscous problem as $\alpha R \to \infty$ and the eigenvalues (5.2) and eigenfunctions (5.3) of the inviscid problem. In principle at least it should be possible to determine analytically which of the viscous modes have inviscid limits in the form of internal gravity waves and which, if any, do not. Unfortunately, the asymptotic analysis of this problem would appear to be substantially more difficult than the one given in §4 for $0 < Ri < \frac{1}{4}$ and such an analysis has not yet been attempted. Some numerical results have been obtained, however, for $Ri = 10$ and they are shown in figure 6. Of the twelve modes shown, ten of them have values of c_r which, as $\alpha R \to \infty$, lie outside the range of $U(z)$ and are in close agreement with (5.2). In this limit, therefore, they can be identified as internal gravity waves and they have been indexed accordingly. For the other two modes, however, we find that

$$(1 \pm c_r + ic_i)\,(\alpha R)^{\frac{1}{3}} \to \begin{Bmatrix} 6{\cdot}513 - 3{\cdot}626i, \\ 8{\cdot}770 - 4{\cdot}944i \end{Bmatrix} \tag{5.8}$$

as $\alpha R \to \infty$. These modes have values of c_r which lie inside the range of $U(z)$ and hence they cannot be identified with the internal gravity waves. The existence of such modes having no inviscid limit throughout the closed interval $|z| \leqslant 1$ would then suggest, by analogy with the corresponding problem for a non-stratified fluid, that the inviscid problem may also possess a continuous spectrum in addition to the discrete spectrum of internal gravity waves. We believe, therefore, that further clarification is needed concerning the relationship between the inviscid initial-value problem on the one hand and the limiting behaviour as $\alpha R \to \infty$ of the discrete spectrum of the viscous problem on the other.

6. Discussion

The results presented in this paper would appear to be reasonably complete when the overall Richardson number lies in the range $0 \leqslant Ri < \frac{1}{4}$. When $Ri = \frac{1}{4}$, the roots of the indicial equation are equal and we then have many of the same difficulties as occur in the study of the Orr–Sommerfeld equation. When $Ri > \frac{1}{4}$, $p_2 = p_1^*$ with $\mathrm{Re}\,(p_1) = \frac{1}{2}$ and some of the approximations which led to (4.16) are no longer permissible. A more detailed study of this last case would be of particular interest in attempting to clarify the relationship between the viscous theory, for which all the modes are discrete, and the inviscid theory, for which there may be a continuous spectrum in addition to the discrete spectrum of internal gravity waves.

Many of the present results were obtained on the assumption that $\alpha = 0$ but αR is finite, and this assumption certainly simplifies some parts of the analysis and reduces the amount of numerical work required. The dependence of the velocity modes on α has been thoroughly examined by Gallagher (1974) and it would be reasonable to expect similar variations in the stratified case. It is perhaps also worth noting that the limiting behaviour of $c_i(\alpha R)^{\frac{1}{3}}$ is independent of α.

There are a number of respects in which the present theory can be generalized to include, for example, more general forms for the basic flow and for the buoyancy. A case of particular interest, which will be considered in part 2, is plane Couette flow with $N^2(z) = z^2$. This is a case where the basic flow is stable in the absence of stratification but is unstable, as Huppert's (1973) inviscid analysis shows, when the stratification is statically stable, contrary to what one might have thought intuitively.

We are grateful to Dr P. G. Drazin for some helpful comments which led to clarification of the discussion given in § 5. The present work was begun while one of us (W.H.R.) was a visiting member of the School of Mathematics, University of Newcastle upon Tyne (from September 1973 to March 1974) and he is grateful to Professor P. H. Roberts for his kind hospitality and interest in the work. The research reported in this paper has been supported in part by the National Science Foundation under grants GP-33131X and MCS75-06499 A01 with the University of Chicago.

Appendix. The zeros of $A_1(z, p)$

In § 4 it was shown that the limiting behaviour of the modes as $\alpha R \to \infty$ with $0 \leqslant Ri < \frac{1}{4}$ could be expressed in terms of the zeros of the generalized Airy function $A_1(z, p)$. This function is defined for unrestricted (complex) values of the parameter p by

$$A_1(z,p) = \frac{1}{2\pi i}\int_{L_1} t^{-p} \exp\left(zt - \tfrac{1}{3}t^3\right) dt, \qquad \text{(A 1)}$$

p	Ri	$-a_1(p)$	$-a_2(p)$	$\mathrm{Im}\,(a_1 e^{-\frac{1}{3}\pi i})$	$\mathrm{Im}\,(a_2 e^{-\frac{1}{3}\pi i})$
−0·50	0·25	1.6172	3·7115	0·8086	1·8558
−0·45	0·2475	1·6823	3·7558	0·8412	1·8779
−9·40	0·24	1·7486	3·7992	0·8743	1·8996
−0·35	0·2275	1·8162	3·8417	0·9081	1·9208
−0·30	0·21	1·8852	3·8829	0·9426	1·9415
−0·25	0·1875	1·9558	3·9227	0·9779	1·9613
−0·20	0·16	2·0281	3·9607	1·0140	1·9803
−0·15	0·1275	2·1022	3·9966	1·0511	1·9983
−0·10	0·09	2·1784	4·0301	1·0892	2·0151
−0·05	0·0475	2·2570	4·0607	1·1285	2·0304
0·00	0·00	2·3381	4·0879	1·1691	2·0440
0·05	—	2·4222	4·1112	1·2111	2·0556
0·10	—	2·5099	4·1297	1·2549	2·0648
0·15	—	2·6016	4·1427	1·3008	2·0713
0·20	—	2·6983	4·1490	1·3492	2·0745
0·25	—	2·8012	4·1473	1·4006	2·0737
0·30	—	2·9123	4·1356	1·4561	2·0678
0·35	—	3·0345	4·1105	1·5173	2·0552
0·40	—	3·1741	4·0658	1·5870	2·0329
0·45	—	3·3465	3·9862	1·6732	1·9931
0·50	0·25	3·7115	—	1·8558	1·8558
0·55	0·2475	$3·7556 \pm 0·3287i$		1·5932	2·1624
0·60	0·24	$3·7986 \pm 0·4708i$		1·4916	2·3070
0·65	0·2275	$3·8404 \pm 0·5836i$		1·4148	2·4257
0·70	0·21	$3·8813 \pm 0·6818i$		1·3502	2·5311
0·75	0·1875	$3·9211 \pm 0·7708i$		1·2930	2·6281
0·80	0·16	$3·9600 \pm 0·8534i$		1·2410	2·7190
0·85	0·1275	$3·9980 \pm 0·9311i$		1·1926	2·8053
0·90	0·09	$4·0351 \pm 1·0051i$		1·1472	2·8880
0·95	0·0475	$4·0714 \pm 1·0759i$		1·1040	2·9675
1·00	0·00	$4·1070 \pm 1·1442i$		1·0626	3·0444

TABLE 1. The first two zeros of $A_1(z, p)$ for $p = -0·50(0·05)1·00$.

where $0 \leqslant \mathrm{ph}\, t < 2\pi$ and L_1 is the usual Airy contour that runs from $\infty e^{\frac{4}{3}\pi i}$ to $\infty e^{\frac{2}{3}\pi i}$. Alternatively, $A_1(z, p)$ can be defined as the solution of the differential equation

$$\left(\frac{d^3}{dz^3} - z\frac{d}{dz} + p - 1\right) y = 0 \tag{A 2}$$

with the initial values

$$A_1(0, p) = e^{-p\pi i} 3^{-\frac{1}{3}(p+2)}/\Gamma(\tfrac{1}{3}(p+2)), \quad A_1'(0, p) = A_1(0, p-1). \tag{A 3}$$

It will be convenient for the present purposes to denote the zeros of $A_1(z, p)$ by $a_s(p)$ $(s = 1, 2, \ldots)$. Thus, if a_s and a_s' denote, as usual, the zeros of $\mathrm{Ai}\,(z)$ and $\mathrm{Ai}'(z)$ respectively, then we have

$$a_s(0) \equiv a_s, \quad a_s(-1) \equiv a_s'. \tag{A 4}$$

Furthermore, for half-integral values of p it is not difficult to show (see, for example, Hughes & Reid 1968) that $A_1(z, p)$ can be expressed in terms of products of $\mathrm{Ai}\,(x)$ and $\mathrm{Ai}'\,(x)$, where $x = 2^{-\frac{2}{3}} z$. Thus, for example, we have

$$A_1(z, \tfrac{3}{2}) = -i 2^{\frac{4}{3}} \pi^{\frac{1}{2}} \{x\, \mathrm{Ai}^2\,(x) - \mathrm{Ai}'^2\,(x)\}, \tag{A 5}$$

from which (4.17) can be obtained by differentiation. Accordingly, for $p = \pm\frac{1}{2}$ we have

$$a_{2s-1}(\tfrac{1}{2}) = a_{2s}(\tfrac{1}{2}) = 2^{\frac{2}{3}}a_s \tag{A 6}$$

and

$$a_{2s-1}(-\tfrac{1}{2}) = 2^{\frac{2}{3}}a_s', \quad a_{2s}(-\tfrac{1}{2}) = 2^{\frac{2}{3}}a_s. \tag{A 7}$$

These results provide a useful check on the numerical values given in table 1, which were obtained by direct integration of (A 2) using the initial values (A 3). The results for $0 < p < \frac{1}{2}$ are not needed for the stability problem but they have been included to illustrate the general behaviour of the zeros as functions of p.

From table 1 it can be seen that the zeros of $A_1(z,p)$ are all real if $p \leqslant \frac{1}{2}$ and that they occur in complex-conjugate pairs if $p > \frac{1}{2}$. More generally, Wasow's (1953) proof that $A_1(z,1)$ has no real zeros can easily be extended to show that $A_1(z,p)$ has no real zeros if $p > \frac{1}{2}$. For $p > \frac{1}{2}$ we have also adopted the conventions that $\mathrm{Im}\{-a_{2s-1}(p)\} > 0$ and that $a_{2s}(p) = a_{2s-1}^*(p)$.

It is also of some interest to obtain asymptotic approximations to the zeros. By a slight modification of the analysis given by Reid (1972) it can be shown that

$$A_1(-z,p) \sim \frac{1}{\Gamma(p)}e^{-p\pi i}z^{p-1} + \pi^{-\frac{1}{2}}e^{-p\pi i}z^{-\frac{1}{4}(2p+1)}\sin(\xi - \tfrac{1}{2}p\pi + \tfrac{1}{4}\pi), \tag{A 8}$$

where $\xi \equiv \frac{2}{3}z^{\frac{3}{2}}$, and this approximation is valid as $z \to \infty$ in the sector $|\mathrm{ph}\, z| < \frac{2}{3}\pi$. For $p < \frac{1}{2}$, the algebraic term in (A 8) can be neglected and we then obtain

$$-a\,(p) \sim [\tfrac{3}{8}\pi(4s+2p-1)]^{\frac{2}{3}} \quad (p < \tfrac{1}{2}). \tag{A 9}$$

This approximation is not uniformly valid, however, as $p \uparrow \frac{1}{2}$. For $p \geqslant \frac{1}{2}$, the algebraic term in (A 8) cannot be neglected. In this case it is more convenient to denote the zeros by $\alpha_s(p)$ and $\alpha_s^*(p)$ $(s = 1, 2, \ldots)$, where, by convention, $\mathrm{Im}\{-\alpha_s(p)\} > 0$. In terms of this notation we have

$$a_{2s-1}(p) = \alpha_s(p), \quad a_{2s}(p) = \alpha_s^*(p). \tag{A 10}$$

It is then not difficult to generalize the result given by Zondek & Thomas (1953) for $p = 1$ to obtain

$$\{-\alpha_s(p)\}^{\frac{3}{2}} \sim \tfrac{3}{8}\pi(8s+2p-3) + i\tfrac{3}{2}\cosh^{-1}\left\{\frac{\pi^{\frac{1}{2}}}{\Gamma(p)}[\tfrac{3}{8}\pi(8s+2p-3)]^{p-\frac{1}{2}}\right\} \quad (p \geqslant \tfrac{1}{2}). \tag{A 11}$$

If, instead, we had obtained the generalization of the simpler result given by Reid (1974) for $p = 1$, then the resulting approximation would not have been uniformly valid as $p \downarrow \frac{1}{2}$.

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